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When do star clusters have unphysical distribution functions?

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Abstract. It is sometimes impossible to construct a relativistic star cluster with the same structure as a given relativistic fluid sphere. The reason is the unphysical nature of the distribution function which is sometimes negative in part of phase-space or sometimes contains non-integrable singularities. A simple test is derived that can be applied to a given fluid sphere to provide a set of sufficient conditions for the distribution function to be unphysical. The test is applied to some fluid spheres and clusters that have been studied in the literature.

1. Introduction

A technique used in the past to construct models of static spherical relativistic star clusters with isotropic velocity distributions was to take a known solution of the Einstein field equations, representing a finite fluid sphere with isotropic pressure, and calculate the isotropic distribution function that gives rise to a cluster that has the same structure as the fluid sphere. The cluster itself was then obtained by solving the field equations. An isotropic distribution function is one that depends only on the conserved energies of the stars along their trajectories, and the stellar rest masses. The mathematical prescription for calculating this distribution function was given by Fackerell (1968). Although the resulting cluster has the same pressure p and density of mass-energy ρ as the fluid sphere, and the same coordinate radius and total mass-energy, it has in general a different rest-mass density and binding energy. This technique was used by Fackerell (1968, 1970) and Ipser (1969) to construct various clusters with polytropic structures.

A problem that can arise with the above process is that there is no guarantee that there exists a physically acceptable isotropic distribution function for a given fluid sphere. A physically acceptable distribution function is positive throughout phase-space and has at most integrable singularities. (See § 2 for further discussion on this point.) For example, Fackerell (1968) found the distribution function corresponding to the Schwarzschild interior solution to be negative over part of phase-space.

It follows automatically that if no physically acceptable isotropic distribution function exists for a particular solution of the field equations with isotropic pressure, then neither does there exist an anisotropic distribution function, i.e. one that depends on the angular momenta of the stars as well as their energies and rest masses. This follows because an anisotropic distribution function always gives rise to a cluster with anisotropic pressure.

The purpose of this paper is to derive a means by which we may distinguish those solutions of the field equations that give rise to unphysical distribution functions, without first calculating the distribution function itself. We derive here a simple test that can be applied to the behaviour of the pressure and density of mass-energy near the surface of the fluid sphere. If the test is satisfied, it provides a sufficient (although not necessary) condition for the corresponding distribution function to be unphysical.

The conditions on p and ρ are derived in § 2, while § 3 applies the results to some clusters that have been studied in the literature.

2. Sufficient conditions for unphysical distribution functions

We use the static spherical line element

$$\mathrm{d}s^2 = e^{\nu}\,\mathrm{d}t^2 - \mathrm{d}^{\lambda}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

in Schwarzschild coordinates. For spherical clusters with isotropic pressure, the density of stars in phase-space is given by a distribution function of two variables $F(m, E)$ where m is the rest mass of a typical star in the cluster and E is the conserved energy along a stellar trajectory as measured at infinity (Fackerell 1968). However, as far as the analysis in this paper is concerned, it is only necessary to consider a distribution function of a single variable $x = E^2/m^2\beta$. This is because all physical quantities in an isotropic cluster can be expressed in terms of the following mass-weighted distribution function (Fackerell 1968):

$$\chi(x) = \tfrac{1}{6}\pi \int_0^{\infty} F(m, m\beta^{1/2}x^{1/2})m^4\,\mathrm{d}m.$$

In terms of $\chi(x)$, p and ρ can be expressed as (Fackerell 1970)

$$p = y^{-2}\int_y^1 \chi(x)x^{-1/2}(x-y)^{3/2}\,\mathrm{d}x \tag{1}$$

and

$$\rho = 3y^{-2}\int_y^1 \chi(x)x^{1/2}(x-y)^{1/2}\,\mathrm{d}x.$$

Here $y = e^{\nu(r)}/\beta$, where $\beta = e^{\nu(R)}$ is the value of e^{ν} at the surface of the cluster $r = R$. To calculate all values of p and ρ in the range $p_c \geqslant p \geqslant 0$ and $\rho_c \geqslant \rho \geqslant 0$ where p_c and ρ_c are the central values of p and ρ respectively, x ranges over the values $y_c \leqslant x \leqslant 1$. Here $y_c = e^{\nu(0)}/\beta$, and the upper limit on x is 1 because the maximum energy that a star can have in a bounded cluster is $E_{\max} = m\beta^{1/2}$. This guarantees that no stars are found at $r > R$ (Ipser 1969).

2.1. *Conditions on the pressure*

When the pressure p is known as a function of y, equation (1) can be solved for $\chi(x)$ in terms of x. Defining $G(y) = y^2p(y)$, the solution is (Fackerell 1968)

$$\chi(x) = (2/3\pi)x^{1/2}(1-x)^{-3/2}\Big(G'(1)\mathscr{G}(1-x) + 2G''(1)(1-x) - 2(1-x)^{3/2}\int_x^1 G'''(y)(y-x)^{-1/2}\,\mathrm{d}y\Big). \tag{2}$$

In equation (2) (and in the following) a prime denotes differentiation with respect to y.

In addition, $\mathscr{G}(1-x)$ is a generalised function defined by Lighthill (1958) as

$$\mathscr{G}(1-x)=2(1-x)^{3/2}\frac{\mathrm{d}}{\mathrm{d}x}[(1-x)^{-1/2}H(1-x)]$$

where $H(1-x)$ is the Heaviside function.

Although it is not possible to completely analyse $\chi(x)$ in equation (2) for arbitrary values of $p(y)$, it is possible to determine the behaviour of $\chi(x)$ near the surface of the cluster, i.e. for $x \lessdot 1$. The reason is that all clusters of physical interest have surfaces that are regular, and thus the pressure admits a Taylor expansion near the surface of the form

$$p(y)=\sum_{n=1}^{\infty} a_n(1-y)^{s+n}. \tag{3}$$

Here the a_n and s are constants with $a_1>0$ to guarantee $p>0$ as $y\to 1$, and $s\geqslant 0$ to guarantee $p(1)=0$.

It is now a simple matter to calculate the corresponding expansion of $\chi(x)$ when p is given by the series (3). The nature of the expression (2) for $\chi(x)$ does, however, necessitate us considering several distinct values of s separately.

2.1.1. $s=0$. In this case it follows from equation (2) that

$$\begin{aligned}\chi(x)=(2/3\pi)x^{1/2}(1-x)^{-3/2}\bigg[&-a_1\mathscr{G}(1-x)-4(2a_1-a_2)(1-x)+24(a_1-2a_2)(1-x)^2\\&+64a_2(1-x)^3+\frac{1}{2}\sum_{n=3}^{\infty}2^n\frac{n!}{(2n-5)!!}(1-x)^{n-1}\bigg(1-\frac{4(n+1)}{2n-3}(1-x)\\&+\frac{4(n+1)(n+2)}{(2n-1)(2n-3)}(1-x)^2\bigg)a_n\bigg].\end{aligned}$$

Since $\mathscr{G}(1-x)\equiv 1$ when $x<1$, $\chi(x)$ thus has the following behaviour for $x\lessdot 1$:

$$\chi(x)\sim-(2/3\pi)x^{1/2}(1-x)^{-3/2}a_1+\mathrm{O}((1-x)^{-1/2})$$

i.e. $\chi(x)<0$ since $a_1>0$. This rules out the value $s=0$ which from equation (3) corresponds to a finite value of $p'(1)$.

2.1.2. $0<s<1$. It follows from equation (3) that $p''(y)$ is infinite at $y=1$, with the result that $G''(1)$ is infinite. Consequently $\chi(x)$ is infinite for all values of x in the range $y_c<x<1$. Note that the term $2G''(1)(1-x)$ in equation (2) cannot be removed by performing an integration by parts on the integral because the term $(y-x)^{-1/2}$ in the integrand is meaningless at $y=x$. Since the physical quantities associated with the cluster p, ρ, and the density of rest-mass energy ρ_0, are all obtained as integrals over the distribution function, it follows that $\chi(x)$ can contain at most isolated integrable singularities such as delta functions. Since $\chi(x)$ in this case is infinite over an extended range of x, the values $0<s<1$ are also ruled out. As mentioned above, this corresponds to an infinite value for $p''(1)$.

2.1.3. $s=1$. Here $G'(1)=0$ and $G''(1)=2a_1$ so that $\chi(x)$ does not involve a generalised function. The relevant expression for $\chi(x)$ is

$$\chi(x)=(2/3\pi)x^{1/2}(1-x)^{-3/2}\bigg[4a_1(1-x)-48a_1(1-x)^2+64a_1(1-x)^3 + \sum_{n=2}^{\infty} 2^n \frac{(n+1)!}{(2n-3)!!}(1-x)^n\bigg(1-\frac{4(n+2)}{2n-1}(1-x) + \frac{4(n+2)(n+3)}{(2n-1)(2n+1)}(1-x)^2\bigg)a_n\bigg].$$

Thus for $x \lesssim 1$:

$$\chi(x) \sim (8/3\pi)x^{1/2}(1-x)^{-1/2}a_1 + \mathrm{O}((1-x)^{1/2}) > 0.$$

The value $s=1$ is therefore not ruled out by the behaviour of $\chi(x)$ near the surface. When $s=1$, $p'(1)=0$, and $p''(1)$ is finite.

2.1.4. $s>1$. In this last case both $G'(1)=0$ and $G''(1)=0$ with the result that

$$\chi(x)=\frac{4}{3}\pi^{-1/2}x^{1/2}\sum_{n=1}^{\infty}\frac{\Gamma(s+n+1)}{\Gamma(s+n-\frac{3}{2})}(1-x)^{s+n-5/2} \times\bigg(1-\frac{4(s+n+1)}{(2s+2n-3)}(1-x)+\frac{4(s+n+1)(s+n+2)}{(2s+2n-1)(2s+2n-3)}(1-x)^2\bigg)a_n.$$

It is readily seen that for $x \lesssim 1$, $\chi(x)$ has the behaviour

$$\chi(x) \sim \frac{4}{3}\pi^{-1/2}\frac{\Gamma(s+2)}{\Gamma(s-\frac{1}{2})}x^{1/2}(1-x)^{s-3/2}a_1 + \mathrm{O}((1-x)^{s-1/2}) > 0,$$

and hence the values $s>0$ are not ruled out either. For $s>1$ both $p'(1)=0$ and $p''(1)=0$.

The above conditions on the derivatives of p with respect to y at the surface also apply to the derivatives with respect to r. This follows from the field equations which give

$$\frac{\mathrm{d}p}{\mathrm{d}r}=\frac{2M}{R^2}\bigg(1-\frac{2M}{R}\bigg)^{-1}p'(1)$$

and

$$\frac{\mathrm{d}^2p}{\mathrm{d}r^2}=\frac{2M^2}{R^3}\bigg(1-\frac{2M}{R}\bigg)^{-2}p''(1)$$

at the surface. Here M is the total mass-energy of the configuration.

2.2. *Conditions on the mass-energy density*

It is of interest to calculate the behaviour of ρ at the surface in the four cases discussed in § 2.1. This is accomplished by using the relation

$$\rho(y)=-p(y)-2yp'(y). \tag{4}$$

It follows immediately from equations (3) and (4) that $\rho(y)$ admits the expansion

$$\rho(y)=\sum_{n=1}^{\infty} a_n[2y(s+n)-(1-y)](1-y)^{s+n-1}$$

so that the dominant term is

$$\rho(y)\sim 2a_1(s+1)(1-y)^s. \tag{5}$$

Consequently, in the two cases ruled out before (§§ 2.1.1, 2), when $s=0$, ρ is finite at the surface, and when $0<s<1$, ρ has an infinite first derivative at the surface. For the smallest allowed value of s, namely $s=1$, ρ has a finite first derivative at $y=1$.

3. Applications

We apply the results of § 2 to some clusters that have been studied in the literature.

3.1. Polytropic clusters

The structure and dynamic stability of polytropic clusters was studied by Ipser (1969) and Fackerell (1970). They are clusters in which p and ρ are related by the polytropic equations

$$p=\alpha\tau\Theta^{n+1} \tag{6}$$

and

$$\rho=\tau\Theta^n+p/(\Gamma_4-1) \tag{7}$$

where α, τ, n, and Γ_4 are constants, and $\Theta(r)$ is a dimensionless function of the radial variable (Tooper 1965). When equations (6) and (7) are satisfied, p is given in terms of y by

$$p(y)=\frac{\alpha\tau}{(\alpha\gamma_4)^{n+1}}(y^{-\gamma_4/2(n+1)}-1)^{n+1} \tag{8}$$

where $\gamma_4=\Gamma_4/(\Gamma_4-1)$ (Ipser 1969). If we rewrite equation (8) in the form

$$p(y)=\frac{\alpha\tau}{(\alpha\gamma_4)^{n+1}}\left(\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{\gamma_4}{2n+2}\right)_l(1-y)^l\right)^{n+1}$$

where $(\gamma_4/(2n+2))_l$ denotes Pochhammer's symbol, we see immediately that $s=n$, and $a_1=\tau/[\alpha^n(2n+2)^{n+1}]$. As a result, only polytropes with $n\geqslant 1$ can have isotropic distribution functions.

3.2. Das–Narlikar clusters

Das and Narlikar (1975) studied the central redshifts of core-envelope configurations using the method of Bondi (1964). One of their aims was to see if the redshifts of quasars could arise in part from the central redshifts of massive star clusters. However, as Das (1976) discovered when he calculated the distribution function, a star cluster model was not possible for these configurations because the distribution function was always negative over part of phase-space. This could have been predicted *a priori* using the results of the present paper.

The configurations studied by Das and Narlikar consist of an isothermal core surrounded by an adiabatically stable envelope with a finite value of ρ at the surface. Since equation (5) then implies $s=0$, it follows that $\chi(x)<0$ for $x \leqslant 1$. Alternatively, the same conclusion is reached by examining the expression for p, which in the envelope is

$$p=\frac{\rho_{\mathrm{s}}}{n+1}\left(y^{-(n+1)/2}-1\right).$$

Here n is a constant and ρ_{s} is the value of ρ at the surface (Das and Narlikar 1975). Since this last expression can be written as

$$p=\frac{\rho_{\mathrm{s}}}{n+1}\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{n+1}{2}\right)_{l}(1-y)^{l}$$

it follows again that $s=0$.

4. Conclusions and discussion

The results of the calculations in § 2 may be summarised as follows. Take a given finite relativistic fluid sphere with isotropic pressure and the following conditions satisfied at the surface $r=R$. The first derivative of the pressure with respect to r is finite or the second derivative is infinite, and equivalently the density of mass-energy is finite or its first derivative is infinite. If these conditions are satisfied there does not exist a star cluster with the same pressure and density as the fluid sphere because the isotropic distribution function is necessarily negative in part of phase-space, or infinite on all of phase-space. Conversely, in order for there to exist a corresponding star cluster, it is necessary for the pressure to vanish quickly enough at the surface to enable its first derivative to vanish there also, and therefore the density must have at most a finite first derivative at the surface.

When the density is finite at the surface the distribution function is negative near the surface for $x \leqslant 1$, as seen in § 2.1.1. This is the case for the $n=0$ polytropes and the Das–Narlikar models discussed in § 3. A question that arises is why does a physically acceptable isotropic distribution function find it impossible to produce a density distribution that drops abruptly to zero at the surface from some finite value? The answer probably lies in the fact that in an isotropic cluster there is no preferred direction for the stellar velocities. This means that (loosely speaking) equal numbers of stars move in all directions at all points inside the cluster including points just below and at the surface. It is probably this fact that prevents a finite discontinuity occurring in the density at the surface, and more generally prevents the density (and pressure) from vanishing there too rapidly.

In contrast, if all the stars move in circular orbits as they do, for example, in the clusters considered by Einstein (1939), there is no reason why the density (and tangential pressure) cannot be finite at the surface. However, these clusters with purely circular orbits exhibit the most extreme form of anisotropy and it would be of interest to examine the degree of anisotropy required for a finite density at the surface to be generated by a physically acceptable distribution function.

In conclusion, it should be emphasised that the conditions derived in § 2 are not necessary for the occurrence of unphysical distribution functions. The distribution function can still be negative in some regions of phase-space for clusters which have

$s \geqslant 1$. Fackerell (1968) found this to be so for polytropic clusters satisfying the power law equation of state $p = k\rho^{1+1/n}$, where k and n are constants (Tooper 1964). For each value of n that Fackerell studied in the range $1{\cdot}0 \leqslant n \leqslant 3{\cdot}0$, $\chi(x)$ was negative for some values of x in the range $y_c < x < 1$, although it was always the case that $\chi(x) > 0$ when $x \leqslant 1$, since $s = n$ for these clusters.

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